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Local Geometry and Dynamical Behavior on Folded Basic Sets

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Abstract We study new phenomena associated with the dynamics of higher dimensional non-invertible, hyperbolic maps f on basic sets of saddle type; the dynamics in this case presents important differences from the case of diffeomorphisms or expanding maps. We show that the stable dimension (i.e. the Hausdorff dimension of the intersection between local stable manifolds and the basic set) and the unstable dimension (similar definition) give a lot of information about the dynamical/ergodic properties of endomorphisms on folded basic sets. We prove a geometric flattening phenomenon associated to the stable dimension, i.e. we show that if the stable dimension is zero at a point, then the fractal Λ must be contained in a submanifold and f is expanding on Λ. We characterize folded attractors and folded repellers, as those basic sets with full unstable dimension, respectively with full stable dimension. We classify possible dynamical behaviors, and establish when is the system (Λ, f, μ) 1-sided or 2-sided Bernoulli for certain equilibrium measures μ on folded basic sets, for a class of perturbation maps.

Keywords Endomorphisms on folded basic sets of saddle type · Stable/unstable dimensions · 1-sided and 2-sided Bernoulli systems · Invariant manifolds · Hyperbolic attractors/repellers

1 Introduction and Outline of the Paper

The dynamics of hyperbolic diffeomorphisms on basic sets has been studied in detail. Strange attractors are examples of basic sets, which combine properties like the existence of SRB measures with a complicated dynamical behaviour ([2, 3, 20, 22]). For one-dimensional maps it is necessary that the map must be non-invertible in order to have exponential separation of orbits (a feature of chaotic behaviour, see [14]). So non-invertible maps on folded basic sets are natural; their dynamics is shown to be different than that of diffeomorphisms.

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In this paper we deal with the different case of a *smooth endomorphism* f (i.e. a smooth map f on a Riemannian manifold which is not necessarily invertible), in higher dimension; moreover f is assumed to be hyperbolic on a basic set Λ. Such basic sets for smooth endomorphisms will also be called *folded basic sets*. As the map f is not necessarily invertible, nor necessarily expanding on Λ, we cannot apply many of the tools from those two cases. Our map is considered in general to have both stable and unstable directions along the basic set. In the non-invertible case there appear some phenomena which are not encountered in the diffeomorphism case, like the fact that the stable dimension (i.e. the Hausdorff dimension of the intersection between local stable manifolds and the basic set Λ) is not always equal to the zero of the pressure of the stable potential. Also we do not have always the continuity of the stable dimension; in [11] were constructed perturbations f_ε of $f(z, w) := (z^2 + c, w^2)$, $(z, w) \in \mathbb{C}^2$, s.t. f_ε is homeomorphism on the respective basic set Λ_ε. In other examples, namely a family of hyperbolic skew product endomorphisms, there appear both Cantor sets of points in fibers with uncountably many prehistories, and also infinitely many points in fibers having only one prehistory (see [9]).

For an endomorphism on a basic set Λ, one has the natural extension $\hat{\Lambda}$, but this space is not a manifold so we loose the differentiability properties. A very important role in determining (or estimating) the stable dimension is played by the number of f-preimages of a point $x \in \Lambda$, that actually remain in Λ. As the set Λ is not necessarily totally invariant, we may not have all the preimages of a point from Λ, to remain in Λ. In [11] we proved estimates in the hyperbolic case relating this number of preimages remaining in Λ, to the stable dimension. Then in [12] we proved that, if a map is conformal on local stable manifolds, and if this number of preimages is constant (i.e. $f|_\Lambda$ is d-to-one), then the stable dimension is equal to the unique zero of the pressure $t \to P(t\Phi^s - \log d)$. We also gave estimates for the stable dimension by using a new concept, that of inverse pressure ([12]). In [13] we extended these results to the case when the function given by the number of preimages from Λ, denoted by $d(\cdot)$, is locally constant; and also to the case when $d(\cdot)$ is just bounded above by a continuous function $\omega(\cdot)$.

Hyperbolic diffeomorphisms on basic sets, together with equilibrium measures of Holder potentials, are conjugated to Bernoulli shifts (see [2]); this holds since for diffeomorphisms there exist Markov partitions of arbitrarily small diameters on the respective basic sets (see [2, 22]). We have a complete classification of 2-sided Bernoulli shifts with the help of the measure-theoretic entropy, given in the papers of Ornstein (see for example [6]). However for *1-sided Bernoulli shifts*, the above classification is no longer true; indeed we must have also that the numbers of preimages of points, is the same, for the two shifts. And still, there are 1-sided Bernoulli shifts with the same measure theoretic entropy and the same index, which are *not* isomorphic (Parry and Walters, [16]). This points to the subtle chaotic nature of non-invertible maps and the significant changes in dynamics that their foldings produce; see also the notion of folding entropy (introduced in [21]) which is adapted to this case. Ruelle (see [19]) proved that an expanding map, with the equilibrium measure of a Holder continuous potential, is isomorphic to a 1-sided Markov chain, due to the existence of Markov partitions in the expanding case. However 1-sided Markov chains are not necessarily isomorphic to 1-sided Bernoulli shifts ([16]). In our case, for non-invertible smooth maps which are not necessarily expanding, we do not have Markov partitions in general, so we cannot code the dynamics on Λ. Difficulties appear also since, a priori there may exist infinitely many local unstable manifolds through a point from Λ; in the non-invertible case we do not have a nice foliation with unstable manifolds like in the diffeomorphism case.

Outline of Main Results First we will give the definition for basic sets of endomorphisms, and for the preimage counting function $d(\cdot)$; we recall also some estimates for the stable dimension. Hyperbolic attractors and repellers are for instance, particular examples of basic sets.

In Theorem 1 we prove a quite surprising *geometric flattening phenomenon* related to the stable dimension, in certain partially conformal higher dimensional cases; this result says that, if the stable dimension is zero at some point of the basic set Λ, then the preimage counting function is locally constant on Λ and Λ is contained in a union of finitely many invariant submanifolds. So the stable dimension (a local metric invariant) influences strongly the *geometry* of the entire fractal set Λ. In case Λ is connected, we will show in Theorem 2 that $f|_\Lambda$ is expanding. And that, if the pair (Λ, f) is endowed with the measure of maximal entropy, then it becomes tree very weakly Bernoulli; this implies that it is a uniform measure-preserving endomorphism, and thus 1-sided Bernoulli by a result from [4]. By perturbing a hyperbolic map f on a connected basic set Λ, we obtain many examples, namely connected hyperbolic basic sets Λ_g corresponding to the perturbations g of f.

In Theorem 3 we show that a smooth non-expanding endomorphism having non-zero stable dimension, *cannot be* 1*-sided Bernoulli* if endowed with a certain equilibrium measure. We make precise also what we mean by repellers (and local repellers), and prove some of their properties related to the stable dimension and to their stability under perturbations. In Theorem 4 and in Theorem 5 we give necessary and sufficient conditions for a folded basic set to be a (local) *repeller*, respectively an *attractor*, in terms of their stable dimension, respectively unstable dimension. In the non-invertible setting, the two cases are not just the reverse of each other. We also give a strict lower bound for the folding entropy of the stable equilibrium measure in Proposition 4.

Then in Theorem 6 we give a *classification* of the dynamical and symbolical behavior for a class of polynomial endomorphisms on their respective basic sets. We obtain in this case relations between the stable dimension and the 1*-sided* (*or* 2*-sided*) *Bernoullicity* of certain equilibrium measures on those basic sets. We give also a number of examples of dynamical behavior on folded basic sets, among which connected non-Anosov hyperbolic repellers.

2 Main Results and Applications

Basic sets for hyperbolic non-invertible maps appear naturally in many instances. For example we mention the horseshoes with overlaps from [1], folded drapes or veils such as maps of type $(x, y) \to ((A - x - B_1 y)x, (A - B_2 x - y)y)$ (for certain values of the parameters given in [3]), various expanding maps ([6, 19]), hyperbolic skew products with overlaps in fibers, polynomial maps which are hyperbolic on locally maximal invariant sets, etc.; see also [17] for ergodic properties of Axiom A endomorphisms.

We work with a smooth (say $\mathcal{C}^2$) map $f : M \to M$ on a Riemannian manifold; this map, which is not necessarily invertible, is called a smooth *endomorphism*. Let us denote by C_f the critical set of f, i.e. the set of points, around which f is not a local diffeomorphism.

Definition 1 An uncountable, compact set Λ is a *basic set* (or *folded basic set*) for the smooth endomorphism f, if there exists a neighbourhood U of Λ s.t. $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(U)$, and if f is topologically transitive on Λ.

Definition 2 A basic set Λ for an endomorphism f is called an *attractor* if there exists a neighbourhood (denoted also by U without loss of generality) such that $f(\bar{U}) \subset U$. A basic set Λ will be called a *repeller* if there exists a neighbourhood U such that $\bar{U} \subset f(U)$.

If Λ is a compact invariant set for f, then one can introduce the *natural extension* $\hat{\Lambda} := \{\hat{x} := (x, x_{-1}, x_{-2}, \dots), f(x_{-i}) = x_{-i+1} \in \Lambda, i \geq 1, x_0 = x\}$, the canonical projection $\pi : \hat{\Lambda} \to \Lambda$, $\pi(\hat{x}) := x$ and the shift homeomorphism $\hat{f} : \hat{\Lambda} \to \hat{\Lambda}$, $\hat{f}(\hat{x}) = (f(x), x, x_{-1}, x_{-2}, \dots), \hat{x} \in \hat{\Lambda}$. The finite sequences of consecutive preimages $(x, x_{-1}, \dots, x_{-n})$ with $f(x_{-i}) = x_{-i+1}, i = 1, \dots, n$ are called *finite prehistories* of length n, and the infinite sequences of consecutive preimages $\hat{x} = (x, x_{-1}, \dots)$ are called *prehistories* (or *full prehistories*). A point $y \in \Lambda$ such that $f^n(y) = x$ for some $n \geq 1$, will be called an n-preimage of x, or an f^n-preimage of x.

In this paper we work with endomorphisms f which are hyperbolic on basic sets ([20]), i.e. there exists a uniform splitting of the tangent bundle over $\hat{\Lambda}$ into a stable direction E^s_x and an unstable direction $E^u_{\hat{x}}$ which depends on a full prehistory $\hat{x}$. The local stable manifold is denoted by $W^s_r(x)$ and the local unstable manifold by $W^u_r(\hat{x})$. The local unstable manifolds may not form an unstable foliation over Λ, and they may intersect each other both in Λ and/or outside Λ. We assume throughout the paper that the critical set *does not intersect* the basic set Λ. Define the *stable potential* $\Phi^s(x) := \log|Df_s(x)|, x \in \Lambda$, where $|Df_s(x)|$ is the norm of the stable derivative $Df|_{E^s_x}$, $x \in \Lambda$; and similarly the *unstable potential* $\Phi^u(\hat{x}) := -\log|Df_u(\hat{x})|, \hat{x} \in \hat{\Lambda}$.

Definition 3 In the above setting, fix a positive r such that $W^s_r(x)$, $W^u_r(\hat{x})$ exist. Then the Hausdorff dimension $HD(W^s_r(x) \cap \Lambda)$ is called the *stable dimension* at $x \in \Lambda$, and $HD(W^u_r(\hat{x}) \cap \Lambda)$ is called the *unstable dimension* at $\hat{x} \in \hat{\Lambda}$.

Unlike in the case of diffeomorphisms on surfaces (or that of conformal diffeomorphisms), when the stable dimension is equal to the unique zero of the pressure $t \to P(t\Phi^s)$ (see [7]), in the endomorphism case the usual Bowen type equation is not always true (counterexamples in [11]).

Definition 4 In the above setting, let Λ be a basic set for $f : M \to M$, s.t. $C_f \cap \Lambda = \emptyset$. We denote by $d(x) := \text{Card}\{y \in \Lambda, f(y) = x\}, x \in \Lambda$ and call it the *preimage counting function* on Λ.

One notices that, as $C_f \cap \Lambda = \emptyset$, we cannot have preimages with multiplicity greater than 1 near Λ. So all the preimages of x are simple. In this case, by the compactness of Λ, it follows that the function $d(\cdot)$ is upper semi-continuous on Λ. We will define later another notion, that of *index* of a point, with respect to an invariant measure supported on Λ ([16]); the two notions are not the same however. The preimage counting function intervenes in estimates for the stable dimension in the non-invertible case ([11–13]). The following theorem combines results from [11, 12]:

Theorem (Bowen-type formula for the stable dimension in the endomorphisms case) *Assume that $f : M \to M$ is a smooth endomorphism which is hyperbolic on a basic set Λ such that $C_f \cap \Lambda = \emptyset$ and f is conformal on stable manifolds. Then if there exist positive integers d_1, d_2 such that $d_1 \leq d(x) \leq d_2, x \in \Lambda$, it follows that $t_{d_2} \leq \delta^s(x) \leq t_{d_1}, \forall x \in \Lambda$, where for any $p > 0$, t_p denotes the unique zero of the pressure function $t \to P(t\Phi^s - \log p)$. In particular if the number of preimages remaining in Λ is constant and equal to d, then $\delta^s(x) = t_d$.*

We prove now one of our main results in this paper, namely that the *stable dimension influences strongly the geometry* of the whole folded basic set:

Theorem 1 *Let $f : M \to M$ be a smooth endomorphism which is hyperbolic on a basic set Λ with $C_f \cap \Lambda = \emptyset$ and such that f is conformal on local stable manifolds. Assume that d is the maximum possible value of $d(\cdot)$ on Λ, and that there exists a point $x \in \Lambda$ where $\delta^s(x) = t_d = 0$. Then it follows that $d(\cdot) \equiv d$ on Λ and there exist a finite number of unstable manifolds whose union contains Λ. In particular if Λ is connected, then there exists an invariant unstable manifold containing Λ, and $f|_\Lambda$ is expanding.*

Proof If d is the maximum value taken by the preimage counting function $d(\cdot)$ on Λ, and if $\delta^s(x) = t_d$, then we showed in [13] that $d(y) = d, \forall y \in \Lambda$; and thus $\delta^s(y) = t_d$, $y \in \Lambda$, from [12]. By definition of t_d, we have

$$P(t_d \Phi^s - \log d) = 0. \tag{1}$$

In the endomorphism case we obtain similarly, as in the diffeomorphism case of [2], estimates for equilibrium measures on Bowen balls. If ϕ is a Holder continuous potential on Λ, there exists a unique equilibrium measure for ϕ, which is denoted by μ_ϕ. This follows from the bijection between f-invariant probabilities on Λ, and $\hat{f}$-invariant probabilities on $\hat{\Lambda}$; μ is an equilibrium measure for ϕ on Λ if and only if its unique $\hat{f}$-invariant lifting to $\hat{\Lambda}$ is an equilibrium measure for $\phi \circ \pi$. For a Holder continuous function ϕ on Λ let us denote by μ_ϕ its unique equilibrium measure. We denote by $B_n(y, \varepsilon) := \{z \in \Lambda,\ d(f^i z, f^i y) < \varepsilon, i = 0, \ldots, n-1\}$ the (n, ε)-Bowen ball centered at y. Then given a Holder potential ϕ on Λ, one can show, similarly as for diffeomorphisms ([2, 22]) and by working in $\hat{\Lambda}$, that there exist constants $A_\varepsilon, B_\varepsilon > 0$ so that, for any $y \in \Lambda$, $n > 0$, we have:

$$A_\varepsilon e^{S_n(\phi) - nP(\phi)} \le \mu_\phi\big(B_n(y, \varepsilon)\big) \le B_\varepsilon e^{S_n(\phi) - nP(\phi))}.$$

Thus from (1) and since $t_d = 0$, we obtain:

$$\frac{A_\varepsilon}{d^n} \le \mu_0\big(B_n(y, \varepsilon)\big) \le \frac{B_\varepsilon}{d^n}, \tag{2}$$

where μ_0 is the measure of maximal entropy for $f|_\Lambda$.

For two quantities depending on $y \in \Lambda$, $n > 0$ we will say that they are *comparable* if their quotient is bounded above and below by two positive numbers, independently on y, n; this is for example the case in (2) for the quantities $\mu_0(B_n(x, \varepsilon))$ and $\frac{1}{d^n}$.

We want to show now that the cardinality of $W^s_r(x) \cap \Lambda$ is finite. Indeed, let us assume that there are at least N different points inside $W^s_r(x) \cap \Lambda$ and denote their set by $F := \{y^1, \ldots, y^N\}$. Let us take also a fixed, small $\varepsilon > 0$. There exists $n = n(N)$ sufficiently large so that any set of type $f^n(B_n(z, \varepsilon)) \cap W^s_r(x)$ is disjoint from any set of type $f^n(B_n(w, \varepsilon)) \cap W^s_r(x)$ if z, w are n-preimages in Λ of y^i, y^j respectively, and $i \ne j$, $1 \le i, j \le N$. But now the inclination of local unstable manifolds with respect to $W^s_r(x)$ is bounded below by some positive constant, since they are transversal to $W^s_r(x)$ and depend uniformly on prehistories from the compact space $\hat{\Lambda}$. This implies that given a point $y \in F$ and an n-preimage $z \in \Lambda$ of y, we have that the union $\bigcup_{\xi \in F, \xi \ne y} \bigcup_{w \in f^{-n}\xi \cap \Lambda} f^n(B_n(w, \varepsilon))$ does not contain the entire set $f^n(B_n(z, \varepsilon))$. This implies that, in the difference set

$$B_n(z, \varepsilon) \setminus \Bigg[\bigcup_{\xi \in F, \xi \ne y} \bigcup_{w \in f^{-n}\xi \cap \Lambda} B_n(w, \varepsilon)\Bigg],$$

there must exist at least $M(N, \varepsilon)$ mutually disjoint Bowen balls of type $B_{n+k(N,\varepsilon)}(\zeta, \varepsilon/l(N, \varepsilon))$, where $k(N, \varepsilon), l(N, \varepsilon)$ are positive integers. We also recall that, since $C_f \cap$

$\Lambda = \emptyset$, there exists a positive constant ε_0 such that $d(z, w) > \varepsilon_0$ if $f(z) = f(w)$ and $z \neq w, z, w \in \Lambda$. Thus if w, z are different n-preimages of the same point from Λ, then $B_n(z, 4\varepsilon) \cap B_n(w, 4\varepsilon) = \emptyset$ if ε is small enough. By applying the estimates from (2) we obtain that there exists a positive constant D_ε, such that

$$1 \geq \mu_s\left(\bigcup_{y\in F} \bigcup_{y_{-n}\in f^{-n}y\cap\Lambda} B_n(y_{-n}, \varepsilon)\right) \geq D_\varepsilon \cdot \sum_{y\in F, z\in f^{-n}y\cap\Lambda} \mu_s\big(B_n(z, \varepsilon)\big) \geq D_\varepsilon A_\varepsilon N d^n \cdot \frac{1}{d^n}.$$

So the number of points in $W^s_r(x) \cap \Lambda$ must be finite and bounded above by $N(\varepsilon)$, for any $x \in \Lambda$. We recall however that any hyperbolic basic set has local product structure, thus the intersection between any local stable manifold $W^s_r(x)$ and any local unstable manifold $W^u_r(\hat{y})$, $x \in \Lambda$, $\hat{y} \in \hat{\Lambda}$ must belong to Λ, for any $r > 0$ small. Hence Λ must be contained in the union of at most finitely many unstable manifolds, each of type $W^u(\hat{x}, T) := \bigcup_{i=0}^T f^i(W^u_r(\hat{x}))$ for some $T > 0$.

From the finiteness of $W^s_r(x) \cap \Lambda$, it follows that there exists a small $\rho > 0$ such that $W^s_\rho(x) \cap \Lambda = \{x\}$, $x \in \Lambda$. Also notice that Λ does not have isolated points, since it was assumed to be uncountable and topologically transitive.

Let us assume now that Λ is connected and contained in the union of the local unstable manifolds $W^u_1 \cup \cdots \cup W^u_N$. Let us consider for example the intersection between W^u_1 and W^u_2. If there would exist a point $y \in \Lambda \cap W^u_1 \setminus W^u_2$, close to $W^u_1 \cap W^u_2$, then $W^s_\rho(y) \cap W^u_2$ would be in Λ from the local product structure (since we work only near points from Λ, the above intersection is actually an intersection between $W^s_\rho(y)$ and $W^u_r(\hat{\zeta})$ for some $\hat{\zeta} \in \hat{\Lambda}$). But this is a contradiction since we saw that $W^s_\rho(y) \cap \Lambda = \{y\}$. So Λ must be contained in intersections of two or more manifolds W^u_i, $i = 1, \ldots, N$. Now if there would exist only two such manifolds W^u_1 and W^u_2, and if $\Lambda \subset W^u_1 \cap W^u_2$, we are done, since it follows that $\Lambda \subset W^u_1$. If not, since we assumed that Λ is connected, there would exist at least another W^u_3 and $\Lambda_{12} := W^u_1 \cap W^u_2$ would intersect $\Lambda_{23} := W^u_2 \cap W^u_3$ in a point y. But then, again from the non-existence of isolated points in Λ, there must exist some point $z \in \Lambda_{23}$ as close as we want to y. Since $z \in \Lambda$ and f is hyperbolic on Λ, we can construct the local stable manifold $W^s_\rho(z)$; as z is very close to $y \in W^u_1$, we will have that $W^s_\rho(z)$ intersects W^u_1 in a point $\xi \in \Lambda$. But then we obtain a contradiction since $W^s_\rho(z) \cap \Lambda$ would contain more than one point.

The other cases of intersections between the unstable manifolds W^u_i are treated similarly.

Thus if Λ is connected, it must be contained in only one unstable manifold W^u, more precisely in the union of finitely many iterations of one local unstable manifold. In particular it follows that $f|_\Lambda$ is an expanding map in this case. □

We now prove that in the previous Theorem, under the assumption that $f|_\Lambda$ is expanding (which happens, as we saw, when Λ is connected) we have that the measure preserving system (Λ, f, μ_0) is 1-sided Bernoulli, where μ_0 is the unique measure of maximal entropy. In order to do so, we need some notions and results from [4]. Let us denote by $\mathcal{T}$ the tree of f-preimages belonging to Λ, where a node at level n is followed at level $n+1$ by its f-preimages in Λ. A $(\mathcal{T}, \Lambda)$-*name* is a function $h : \mathcal{T} \to \Lambda$; h is called *tree adapted* if for any node v and different f-preimages z, w of v, we have $h(vz) \neq h(vw)$. Given a point $x \in \Lambda$, we shall denote by $\mathcal{T}_x$ the $(\mathcal{T}, \Lambda)$-name represented by the tree of f-preimages of x from Λ. Let now a function $g : \Lambda \to \Lambda$ and $\mathcal{G}$ the σ-algebra on Λ generated by the pullback of the σ-algebra $\mathcal{B}$ of borelians of Λ. We shall say that g *generates* if $\bigvee_i g^{-i}(\mathcal{G}) = \mathcal{B}$. We denote also by $\mathcal{A}$ the collection of all bijections of the nodes of $\mathcal{T}$ that preserve the tree structure, and by $\mathcal{A}_n$ the set of bijections of the set of nodes up to level n, preserving also the tree structure. Then for any $n > 1$, we can define a metric on the space of $(\mathcal{T}, \Lambda)$-names by

$$t_n(g,h) := \inf_{A\in\mathcal{A}_n} \frac{1}{n} \sum_{0<|v|\le n} \frac{1}{d^{|v|}} d\big(h(v), g(Av)\big).$$

Then we shall say as in [4], that the uniform measure preserving system (Λ, f, μ) and the tree adapted map $g : \Lambda \to \Lambda$ are *tree very weakly Bernoulli* if for any $\varepsilon > 0$ and all n large enough, there exists a set $G(\varepsilon, n)$ with $\mu(G(\varepsilon, n)) > 1-\varepsilon$, such that $t_n(\mathcal{T}_z, \mathcal{T}_w) < \varepsilon$, $\forall z, w \in G(\varepsilon, n)$.

Theorem 2 *Assume that Λ is a hyperbolic basic set for a smooth endomorphism f, such that $f|_\Lambda$ is d-to-1, $t_d = 0$ and $f|_\Lambda$ is expanding. Then (Λ, f, μ_0) is 1-sided Bernoulli, where μ_0 is the unique measure of maximal entropy.*

Proof We know that, if $f|_\Lambda$ is expanding, then (Λ, f, μ_ϕ) is isomorphic to a 1-sided Markov chain from [19], where μ_ϕ is the equilibrium measure of an arbitrary Holder potential ϕ. However, as was shown in [16], not all 1-sided Markov chains are isomorphic to 1-sided Bernoulli shifts (unlike the 2-sided Markov chains, which are indeed isomorphic to 2-sided Bernoulli chains, see [6]); this is an important difference between the invertible and non-invertible cases. For the non-invertible case one has to apply more sophisticated results; we shall use results of Hoffman and Rudolph ([4]) about a necessary and sufficient condition which guarantees 1-sided Bernoullicity for certain uniform endomorphisms.

If $f|_\Lambda$ is expanding and d-to-1, we know that μ_0 is the limit of a sequence of measures of type $\mu_n^x := \frac{1}{d^n}\sum_{y\in f^{-n}x\cap\Lambda}\delta_y$ ([6, 19]). This implies that

$$\mu_0\big(f(A)\big) = d\mu_0(A),$$

for any borelian set A so that $f|_A$ is injective. So the Jacobian of the measure μ_0 ([15]) is constant and equal to d, which implies that the conditional probabilities of the preimages of μ_0-almost all points from Λ are the same, namely $\frac{1}{d}$ ([15]). On the other hand, if μ_0 is a measure of maximal entropy and $t_d = 0$, we have that $P(-\log d) = 0$, hence $h_{top}(f|_\Lambda) = \log d$. Thus $h_{\mu_0} = \log d$, the conditional probabilities of the preimages are all $\frac{1}{d}$, and $f|_\Lambda$ is d-to-1. Therefore (Λ, f, μ_0) is a *uniform measure preserving endomorphism*, as defined in [4].

We now use the fact that $f|_\Lambda$ is expanding and open (since $f|_\Lambda$ is d-to-1). It is easy to show then that $f|_\Lambda$ is topologically exact. So for any $\varepsilon > 0$ small there exists some positive integer N so that, given any $y, z \in \Lambda$ and any N-preimage $y_{-N} \in \Lambda$ of y, there exists an N-preimage $z_{-N} \in \Lambda$ of z, such that

$$d(y_{-N}, z_{-N}) < \varepsilon. \tag{3}$$

From the fact that f is uniformly expanding on Λ, it follows that N depends only on ε. As the *generating* function we will take the identity $\mathrm{Id} : \Lambda \to \Lambda$ which obviously generates the σ-algebra of borelians on Λ. From (3), and the fact that local inverse iterates of f contract distances, we infer that given any points $y, z \in \Lambda$, there exists $N = N(\varepsilon)$ such that for any $n > N$, and any n-preimage $y_{-n} \in \Lambda$ of y, there exists a unique n-preimage z_{-n} of z, so that $z_{-n} \in B_n(y_{-n}, \varepsilon)$; and vice-versa, for any n-preimage $z_{-n} \in \Lambda$ of z, there is a unique

n-preimage $y_{-n} \in \Lambda$ of y with $y_{-n} \in B_n(z_{-n}, \varepsilon)$. Therefore for any $\varepsilon > 0$, there exists $N(\varepsilon)$ so that we have:

$$t_n(\mathcal{T}_y, \mathcal{T}_z) < C\varepsilon, \quad \forall y, z \in \Lambda,\ n > N(\varepsilon), \tag{4}$$

where $C > 0$ is a constant, independent of ε, n, y, z (C depends only on the minimum expansion coefficient of f on Λ). So in our case the set $G(\varepsilon, n)$ from the definition of tree very weakly Bernoulli, is the whole Λ. Thus the measure preserving uniform endomophism (Λ, f, μ_0) and the generating function $Id : \Lambda \to \Lambda$ are tree very weakly Bernoulli. In conclusion it follows from [4] that (Λ, f, μ_0) is 1-sided Bernoulli. □

We look now at the opposite case, when the stable dimension is positive. We prove that an endomorphism with positive stable dimension at some point, cannot be 1-sided Bernoulli if endowed with a certain equilibrium measure.

Theorem 3 *Let f be a smooth endomorphism, which is hyperbolic on a basic set Λ, such that $\Lambda \cap C_f = \emptyset$ and f is conformal on stable manifolds. Assume that there exists a point $x \in \Lambda$ with $\delta^s(x) > 0$, and denote by μ_s the equilibrium measure of the potential $\delta^s(x) \cdot \Phi^s(\cdot)$. Then the measure preserving system (Λ, f, μ_s) cannot be 1-sided Bernoulli.*

Proof We assumed that $\delta^s(x) > 0$. As $\delta^s(x) \cdot \Phi^s$ is a Holder continuous potential on Λ, it follows that it has a unique equilibrium measure μ_s. From the definition of equilibrium measures:

$$P\big(\delta^s(x)\Phi^s\big) = h_{\mu_s} + \delta^s(x) \cdot \int_\Lambda \Phi^s(y)\, d\mu_s(y). \tag{5}$$

Assume now that (Λ, f, μ_s) is isomorphic to the 1-sided Bernoulli shift $(\Sigma_d^+, \sigma, \rho_{\mathbf{p}})$, where $\rho_{\mathbf{p}}$ is the probability measure induced on Σ_d^+ by the probability vector $\mathbf{p}$.

We recall the notion of *index* for a measure preserving endomorphism $f : (X, \mu) \to (X, \mu)$ on a Lebesgue space ([16]). As (X, μ) is Lebesgue, it follows that the partition into preimage sets of points, $\{f^{-1}(x),\ x \in \Lambda\}$ is measurable and thus one can form the canonical system of conditional measures μ_z for μ-almost all $z \in \Lambda$ ([18]).

If $f : X \to X$ is assumed to be at most countable-to-one, we can define the index:

$$\mathrm{ind}_{f,\mu}(z) := \mathrm{Card}(\mathrm{supp}\, \mu_z), \qquad \text{for } \mu\text{-almost all } z \in \Lambda.$$

The index is defined almost everywhere, it is a measurable function and it is preserved by isomorphisms of measure preserving systems ([16]). Therefore in our case, if (Λ, f, μ_s) is isomorphic to $(\Sigma_d^+, \sigma, \rho_{\mathbf{p}})$, then the index ind_{f,μ_s} is equal to d μ_s-almost everywhere. From definition we know that $d = \mathrm{ind}_{f,\mu_s}(y) \le \mathrm{Card}(f^{-1}(y) \cap \Lambda)$, for μ_s-almost all $y \in \Lambda$. Now any non-empty open set must contain Bowen balls; so by using the estimates for the μ_s-measure of Bowen balls, we obtain that the μ_s-measure of any open non-empty set is strictly positive. Thus recalling the notion of preimage counting function from Definition 4, we obtain that:

$$d(y) \ge d, \quad \text{for } y \text{ in a dense set in } \Lambda. \tag{6}$$

This implies that

$$\delta^s(y) \le t_d, \quad y \in \Lambda$$

by a result from [12]; here again t_d is the unique zero of the pressure function $t \to P(t\Phi^s - \log d)$. Indeed in [12], when covering Λ with Bowen balls, one can take the centers of all these balls to be in the respective dense subset; this implies the last displayed inequality for any point $y \in \Lambda$. Therefore we have that $P(\delta^s(x)\Phi^s - \log d) \geq 0$; but since μ_s is the equilibrium measure of the potential $\delta^s(x)\Phi^s$ we obtain:

$$h_{\mu_s} + \delta^s(x) \cdot \int_\Lambda \Phi^s(y)\, d\mu_s(y) \geq \log d.$$

Recalling however that $\delta^s(x) > 0$ and that $\Phi^s < 0$ on the compact set Λ, we have:

$$h_{\mu_s} \geq \log d - \delta^s(x) \cdot \int_\Lambda \Phi^s(y)\, d\mu_s(y) > \log d. \tag{7}$$

But since we assumed the existence of an isomorphism between (Λ, f, μ_s) and $(\Sigma_d^+, \sigma, \rho_{\mathbf{p}})$, we should have $h_{\mu_s} = h_{\rho_{\mathbf{p}}}$; hence from (7), it follows that $h_{\rho_{\mathbf{p}}} > \log d$. However by using the Variational Principle for entropy we obtain $h_{\rho_{\mathbf{p}}} \leq h_{top}(\sigma) = \log d$. This gives a contradiction. Therefore the measure preserving system (Λ, f, μ_s) cannot be 1-sided Bernoulli. □

We study now a class of basic sets on which the endomorphism is locally constant-to-one, namely repellers (cf. Definition 2). They will be proved to be stable under perturbations, giving thus plenty of examples, obtained from simpler ones.

Proposition 1 *If Λ is a repeller for f, then the preimage counting function $d(\cdot)$ is locally constant on Λ. Also if Λ is a hyperbolic repeller, then the local stable manifolds are contained in Λ.*

Proof It is enough to prove that $f^{-1}(\Lambda) \cap U = \Lambda$ for the neighbourhood U from Definition 2. Thus let us take a point $x \in \Lambda$ and $y \in U$ with $f(y) = x$. Since $y \in U$ and $\bar{U} \subset f(U)$, there will exist a preimage $y_{-1} \in U$ of y, then a preimage $y_{-2} \in U$ of y_{-1}, etc. We obtain thus a full prehistory of y belonging to U. And from the fact that $x = f(y) \in \Lambda$, we see that $f^i(y) \in U, i \geq 0$. Now since $\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(U)$, it follows that $y \in \Lambda$. Recalling also that $C_f \cap \Lambda = \emptyset$ we have that $d(\cdot)$ is locally constant.

Now let us take a local stable manifold $W_r^s(x)$ and let $y \in W_r^s(x)$. Then if $r > 0$ is sufficiently small, we have that $W_r^s(x) \subset U$ and thus y has a full prehistory belonging to U. Meanwhile, $f^i y \in U, i \geq 0$ since $y \in W_r^s(x)$. From the definition of basic set, it follows that $y \in \Lambda$, hence $W_r^s(x) \subset \Lambda$. □

Proposition 2 *If Λ is a hyperbolic repeller for a smooth endomorphism f, and if g is a close $\mathcal{C}^1$-perturbation of f near Λ, it follows that g has a hyperbolic repeller Λ_g close to Λ. Moreover if Λ is connected, then Λ_g is also connected.*

Proof If g is a perturbation of f, then there exists a conjugating homeomorphism $\Phi_g : \hat{\Lambda} \to \hat{\Lambda}_g$ commuting with $\hat{f}$ and $\hat{g}$ ([8, 20]). Since Λ is a repeller, there exists a neighbourhood U of Λ such that $\bar{U} \subset f(U)$. Now if g is a close perturbation of f, it follows that $\bar{U} \subset g(U)$, so $\Lambda_g := \bigcap_{n \in \mathbb{Z}} g^n(U)$ is a repeller for g. It can be shown easily that g is hyperbolic on Λ_g too.

For the last statement, assume that Λ is in addition connected. Then $\hat{\Lambda}$ is connected too, and hence $\hat{\Lambda}_g$ is connected; thus $\Lambda_g = \pi(\hat{\Lambda}_g)$ is connected as well. This connectedness property holds for any basic set Λ, not only for repellers. □

In conclusion for a hyperbolic repeller Λ, the stable dimension at any point is equal to the real dimension of the stable manifold.

We introduced in [10] the notion of *local repeller*, which is a basic set Λ which contains some local stable manifold. If $f|_\Lambda$ is open and Λ is not a repeller, then Λ is not a local repeller either ([10]). Let us denote the real dimension of the stable tangent subspaces by d_s, and the real dimension of the unstable tangent subspaces by d_u; they are non-negative integers.

Theorem 4 *Let Λ be a basic set for a hyperbolic endomorphism f such that $C_f \cap \Lambda = \emptyset$ and f is conformal on local stable manifolds. Then Λ is a local repeller if and only if there exists $x \in \Lambda$ with $\delta^s(x) = d_s$.*

Proof If Λ is a local repeller there must exist local stable manifolds contained in Λ, thus there is some point $x \in \Lambda$ with $\delta^s(x) = d_s$.

The converse is proved in [10] for the case $d_s = 2$, and can be extended to the general case in the same way (if f is conformal on local stable manifolds). □

Proposition 3 *Let Λ be a hyperbolic basic set for a smooth endomorphism f, such that $C_f \cap \Lambda = \emptyset$; assume that f is conformal on stable manifolds (which are supposed to have real dimension d_s). Assume also that the stable dimension is strictly less than d_s at all points of Λ. Then for any $\mathcal{C}^1$ perturbation f_ε of f, there exists a hyperbolic basic set Λ_ε of f_ε, close to Λ. Moreover the stable dimension is strictly less than d_s at all points of Λ_ε, hence Λ_ε cannot be a repeller for f_ε.*

Proof If the stable dimension at any point of Λ is strictly less than d_s, it follows that Λ cannot be a local repeller. Therefore if f_ε is a $\mathcal{C}^1$ perturbation of f, it follows from [10] that Λ_ε cannot be a local repeller for f_ε. Then applying Theorem 1 of [10] (or an easy generalization of that theorem to the general case of stable dimension d_s), we obtain that the stable dimension $\delta^s(y) < d_s$ for any point $y \in \Lambda_\varepsilon$. Therefore Λ_ε cannot be a repeller for f_ε. □

Ruelle studied in [21] a notion of *folding entropy*. In our setting we assume Λ is a basic set for f and μ is an f-invariant probability measure on Λ. Then the folding entropy of μ is defined as the conditional entropy

$$F_\mu(f) := H_\mu(\epsilon | f^{-1}\epsilon),$$

where ϵ is the partition of Λ into single points.

Proposition 4 *Assume that the basic set Λ is not a local repeller, $C_f \cap \Lambda = \emptyset$ and f is conformal on local stable manifolds. Then for any f-invariant probability measure μ on Λ, it follows that*

$$h_\mu < F_\mu(f) - \int_\Lambda d_s \Phi^s(x)\, d\mu(x).$$

In particular $P(d_s\Phi^s) < F_{\mu_s}(f)$, where μ_s is the equilibrium measure of $d_s\Phi^s$.

Proof From the conformality of f on local stable manifolds, it follows that $\log|\det Df_s(x)| = d_s \log|Df_s(x)|$, $x \in \Lambda$, where we recall that d_s is the real dimension of the stable tangent subspaces. From [5] we have the first inequality above, with less or equal sign, and also that equality happens if and only if μ has absolutely continuous conditional measures on the stable manifolds.

But we saw in [10] that, if Λ is not a local repeller, then $\delta^s(x) < d_s$, $x \in \Lambda$, where d_s denotes the common real dimension of the stable manifolds. Now recall that the conditional measures on local stable manifolds (their existence is proved in [5]) are probability measures supported on stable intersections of type $W^s_r(x) \cap \Lambda$ for μ-almost all $x \in \Lambda$.

Since $HD(W^s_r(x) \cap \Lambda) < d_s$, $x \in \Lambda$ we obtain that these conditional measures of μ on local stable manifolds cannot be absolutely continuous with respect to the induced Lebesgue measures. We also notice that if λ_i^- denote a negative Lyapunov exponent of multiplicity $m_i(x)$, then $\int \sum_i \lambda_i(x)^- m_i(x)\, d\mu(x) = d_s \int_\Lambda \Phi^s(x)\, d\mu(x)$, from the Birkhoff Ergodic Theorem. So the first inequality from the statement is indeed strict. The second inequality follows from the Variational Principle for pressure. □

As examples of hyperbolic repellers, one can take hyperbolic toral endomorphisms; in that case Λ is the entire torus. Other non-Anosov examples can be obtained by perturbations of hyperbolic product maps. For instance we can take $f : \mathbb{P}^1\mathbb{C} \times \mathbb{T}^2 \to \mathbb{P}^1\mathbb{C} \times \mathbb{T}^2$, $f([z_0 : z_1], (x, y)) = ([z_0^2 : z_1^2], f_A(x, y))$, where f_A is the toral endomorphism induced by a 2×2 integer-valued matrix A; assume also that A is hyperbolic, i.e. it has one eigenvalue of absolute value strictly less than 1, and another eigenvalue with absolute value strictly larger than 1. Then the basic set $\Lambda := S^1 \times \mathbb{T}^2$ is a connected hyperbolic repeller for f. We can take then a perturbation f_ε of f, which will present a *hyperbolic repeller* Λ_ε, close to Λ. From Proposition 2 it follows that Λ_ε is also connected.

We saw that hyperbolic repellers are characterized (in certain cases) with the help of the stable dimension, and one can use the unstable dimension to determine whether a hyperbolic basic set is an attractor. The two cases are however very different in the non-invertible setting, and imply different methods. Indeed in [10] we used the inverse pressure to get the result for local repellers, whereas for the attractor case we use results from [8] and [17].

Theorem 5 *Let Λ a hyperbolic basic set for a smooth endomorphism $f : M \to M$ defined on a Riemannian manifold. Assume that f is conformal on local unstable manifolds* (*which are supposed to have real dimension d_u*). *Then Λ is an attractor for f if and only if there exists $\hat{x} \in \hat{\Lambda}$ with $\delta^u(\hat{x}) = d_u$.*

Proof From [8] we know that if f is conformal on local unstable manifolds, then the unstable dimension $\delta^u(\hat{x})$ is equal to the unique zero of the pressure function $t \to P(t\Phi^u)$, where $\Phi^u(\hat{x}) := -\log|Df_u(\hat{x})|$, $\hat{x} \in \hat{\Lambda}$. On the other hand, it was shown in [17] that the basic set Λ is an attractor for f if and only if $P(-\log|\det Df_u(\cdot)|) = 0$.

Since f is conformal on local unstable manifolds, and since the real dimension of the unstable tangent subspaces is d_u, we obtain $-\log|\det Df_u(\hat{x})| = -d_u \log|Df_u(\hat{x})|$, $\hat{x} \in \hat{\Lambda}$. In conclusion Λ is an attractor if and only if $P(d_u\Phi^u) = 0$, which is equivalent to $\delta^u(\hat{x}) = d_u$, $\hat{x} \in \hat{\Lambda}$. □

We now give a *Classification Theorem* for the possible dynamical behaviors that a certain class of smooth maps may present on their respective basic sets:

Theorem 6 *For some small $|c|$, $c \in \mathbb{C} \setminus \{0\}$, let us consider the polynomial map $f(z, w) = (z^2 + c, w^2)$, $(z, w) \in \mathbb{C}^2$. Let also a polynomial f_ε which is a smooth perturbation of f and let Λ_ε be the corresponding basic set of f_ε close to the set $\Lambda := \{p_c\} \times S^1$* (*where p_c*

is the fixed attracting point of $z \to z^2 + c$). Then we may have exactly one of the following possibilities:

(a) *There exists a point $x \in \Lambda_\varepsilon$ where $\delta^s(x) = 0$. Then there exists a manifold W such that $\Lambda_\varepsilon \subset W$, $f_\varepsilon|_{\Lambda_\varepsilon}$ is expanding and $f_\varepsilon|_{\Lambda_\varepsilon}$ is 2-to-1. In this case the stable dimension is 0 at any point from Λ_ε, and the measure preserving system $(\Lambda_\varepsilon, f_\varepsilon, \mu_{0,\varepsilon})$ is 1-sided Bernoulli (where $\mu_{0,\varepsilon}$ is the unique measure of maximal entropy for $f_\varepsilon|_{\Lambda_\varepsilon}$).*

(b) *There exists a point $x \in \Lambda_\varepsilon$ with $0 < \delta^s(x) < 2$. Then the stable dimension is positive at any point of Λ_ε, and the measure preserving system $(\Lambda_\varepsilon, f_\varepsilon, \mu_{s,\varepsilon})$ cannot be 1-sided Bernoulli, where $\mu_{s,\varepsilon}$ is the equilibrium measure of the potential $\delta^s(x)\Phi^s_\varepsilon$. We have two subcases*:

   (b1) *$f_\varepsilon|_{\Lambda_\varepsilon}$ is a homeomorphism, and in this case the measure preserving system $(\Lambda_\varepsilon, f_\varepsilon, \mu_\phi)$ is 2-sided Bernoulli for any Holder continuous potential ϕ, where μ_ϕ is the equilibrium measure of ϕ.*

   (b2) *there exist both points with only one f_ε-preimage in Λ_ε, as well as points with two f_ε-preimages in Λ_ε; the set of points with one f_ε-preimage in Λ_ε has non-empty interior.*

Proof (a) If f_ε is a smooth perturbation of f, then it has a hyperbolic basic set Λ_ε and we are in the situation of Theorem 1. Moreover notice that, from the existence of a homeomorphism between $\hat\Lambda$ and $\hat\Lambda_\varepsilon$, it follows that Λ_ε is connected. So we have that, if there exists a point from Λ_ε with zero stable dimension, then Λ_ε is contained in an unstable manifold, $f|_{\Lambda_\varepsilon}$ is expanding and $f_\varepsilon|_{\Lambda_\varepsilon}$ is 2-to-1. Now notice that from the existence of the conjugation between $(\hat\Lambda, \hat f)$ and $(\hat\Lambda_\varepsilon, \hat f_\varepsilon)$, we have $h_{top}(f_\varepsilon|_{\Lambda_\varepsilon}) = \log 2$; hence $t_2 = 0$. Thus if $f_\varepsilon|_{\Lambda_\varepsilon}$ is expanding and 2-to-1, we obtain from Theorem 2 that $(\Lambda_\varepsilon, f_\varepsilon, \mu_0)$ is 1-sided Bernoulli.

(b) This follows from Theorem 3. In case $f_\varepsilon|_{\Lambda_\varepsilon}$ is 1-to-1, it follows that it is a homeomorphism, thus a diffeomorphism near Λ_ε. We can calculate then the stable dimension as for a diffeomorphism, from the usual Bowen type equation, as the unique zero of the pressure of the stable potential.

If the set of points with only one f_ε-preimage in Λ_ε had empty interior, there would exist a dense set of points in Λ_ε, each one having two f_ε-preimages in Λ_ε. But since the preimage counting function is upper semi-continuous, we conclude that every point from Λ_ε would have two f_ε-preimages. So we are back in case (a). Thus we either have subcase (b1) or subcase (b2). Also from Proposition 3 we see immediately that $\delta^s(x) < 2$, $\forall x \in \Lambda_\varepsilon$. □

In the end, we make the observation that there may exist cases of hyperbolic endomorphisms which are homeomorphisms on their basic sets, for instance the polynomial maps of [11],

$$f_\varepsilon(z, w) = \left(z^2 + a\varepsilon z + b\varepsilon w + c + d\varepsilon zw + e\varepsilon w^2, w^2\right), \quad (z, w) \in \mathbb{C}^2,$$

where $b \neq 0$, $0 \neq |c| < c(a, b, d, e)$, $0 < \varepsilon < \varepsilon(a, b, c, d, e)$. This map was shown to be hyperbolic on its respective basic set $\Lambda_\varepsilon := \bigcap_{n\in\mathbb{Z}} f^n(U)$, where U is some small neighbourhood of $\Lambda := \{p_c\} \times S^1$, where p_c is the unique fixed attracting point of $z \to z^2 + c$. Also we proved that $f_\varepsilon|_{\Lambda_\varepsilon}$ is a homeomorphism (case (b1) of Theorem 6). Thus the stable dimension in this case can be computed from the Bowen equation $\delta^s(x) = t_s$, where t_s is the unique zero of the function $t \to P(t\Phi^s_\varepsilon)$.

One can have also the basic set contained in an unstable manifold (case (a) of Theorem 6). For example the map $f_\varepsilon(z, w) = (z^2 + \varepsilon w^4, w^2)$, $(z, w) \in \mathbb{C}^2$ has a basic set Λ_ε close to

$\{0\} \times S^1$. Moreover its basic set is contained in the submanifold

$$W := \left\{(z, w) \in \mathbb{C}^2,\ z = \alpha \cdot w^2)\right\},$$

where $\alpha = \frac{1-\sqrt{1-4\varepsilon}}{2}$. In fact we have in this case $f(W) = W = f^{-1}(W)$. In this case f_ε is expanding and 2-to-1 on Λ_ε, and the stable dimension is everywhere equal to 0.

It is more difficult to give examples where the number of preimages varies. Still if f and Λ are in this case, we can take approximating continuous functions $\omega(\cdot)$ so that $d(y) \le \omega(y)$, $y \in \Lambda$. For example, if $d(\cdot)$ takes only two values on Λ, namely $d(x) = d_1$, $x \in \Lambda_1$ and $d(y) = d_2$, $y \in \Lambda_2$, and if $\mathring{\Lambda}_1 \ne \emptyset$, we may take a continuous function $\omega(\cdot)$ so that $\omega \equiv d_2$ on a neighbourhood V_2 of Λ_2, $\omega \equiv d_1$ on some open set V_1 with $\bar{V}_1 \subset \mathring{\Lambda}_1$ and $d_1 \le \omega(x) \le d_2$ for other points $x \in \Lambda$. Then from [13], we know that $t_{d_1} \ge \delta^s(x) \ge t_\omega$, where t_ω is the unique zero of the pressure function $t \to P(t\Phi^s - \log \omega)$.

In the general case $d(\cdot)$ may take only finitely many values on the compact set Λ and we can proceed in a similar fashion with approximating functions ω in order to get estimates for the stable dimension.

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